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Aequationes Mathematicae

Stability of generalized Cauchy equations

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Dedicated to Professor János Aczél on his 90th Birthday

Abstract. We investigate the stability of the functional equation

$$f(xy) = g(x)h(y) + k(y)$$

on amenable semigroups. This equation is a common generalization of two Pexider equations stemming from Cauchy's additive and multiplicative functional equations, and it is a simple case of the Levi-Civita equation.

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Keywords. Cauchy equations, stability in the sense of Hyers–Ulam, Pexider equations, Levi-Civita functional equation.

1. Introduction

A common generalization of Pexider's equations

$$f(xy) = g(x) + k(y)$$

and

$$f(xy) = g(x)h(y)$$

is

$$f(xy) = g(x)h(y) + k(y). \tag{1.1}$$

For real functions this equation has already been treated in J. Aczél's fundamental monograph [1], where the composition xy means addition of real numbers (therefore we have $f(x+y) = g(x)h(y) + k(y)$ on p. 120 of [1]). Then chapter 15 of the book [3] is devoted to equation (1.1). By referring to J. Aczél [2] , it is solved under the assumptions that $f, g, h, k \colon S \to \mathbb{F}$, where S is an abelian groupoid with neutral element and $\mathbb{F}$ is a field. More precisely, when solving Exercise 1 on p. 250 of [3], it turns out that f, g, h, k satisfy (1.1) if and only if they have one of the following forms (where $b, c, u, v \in \mathbb{F}$):

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1. $f(x) = b$, g arbitrary, $h = 0$, $k(x) = b$.
2. $f(x) = b$, $g(x) = c$, h arbitrary, $k(x) = b - ch(x)$.
3. $f(x) = v(a(x) + b)$, $g(x) = a(x) + b - c$, $h(x) = v$, $k(x) = v(a(x) + c)$, where $a\colon S \to \mathbb{F}$ solves the additive Cauchy equation
$$a(xy) = a(x) + a(y), \quad x, y \in S.$$
4. $f(x) = v(ce(x) + b)$, $g(x) = ce(x) + u$, $h(x) = ve(x)$, $k(x) = v(b - ue(x))$, where $e\colon S \to \mathbb{F}$ solves the multiplicative Cauchy equation
$$e(xy) = e(x)e(y), \quad x, y \in S.$$

The just given four types of functions $f, g, h, k\colon S \to \mathbb{F}$ are also solutions of (1.1) in *arbitrary* (not necessarily commutative) groupoids S.

Equation (1.1) is a special case of the Levi-Civita functional equation
$$f(xy) = \sum_{i=1}^{n} g_i(x)h_i(y).$$
For information concerning the solutions of this equation see, for example, [5–10] and [12]. Recently, solutions of (1.1), on various non-abelian groups, have been examined in [4].

In the present paper we investigate the stability in the sense of Hyers-Ulam of equation (1.1) on amenable semigroups. Let us recall that a semigroup S is called right amenable if there exists a right invariant mean on the space $\mathcal{B}(S)$ of all bounded complex-valued functions defined on S. By a right invariant mean we understand a linear functional M satisfying
$$M(\overline{f}) = \overline{M(f)}, \;\; f \in \mathcal{B}(S),$$
$$\inf_{s\in S} f(s) \le M(f) \le \sup_{s\in S} f(s), \;\; \text{for all real-valued } \; f \in \mathcal{B}(S),$$
and
$$M(f_x) = M(f), \;\; f \in \mathcal{B}(S), \; x \in S,$$
where $f_x(s) := f(sx)$, $s \in S$. It is easily seen that the linear functional M has the properties
$$\|M\| \le 2 \tag{1.2}$$
and $M(c) = c$, for all complex numbers c. Moreover, for convenience, we will write $M_s(f(s))$ instead of $M(f)$.

In paper [11] the stability of the functional equation
$$u(Lx) = \alpha(L)u(x) + \beta(L)$$
is under consideration, where X is a set, $\mathcal{L}$ is an amenable group of self-mappings of X, $\mathbb{K}$ is the field of real or complex numbers, $u\colon X \to \mathbb{K}$, $\alpha, \beta\colon \mathcal{L} \to \mathbb{K}$.

The main result of our paper is contained in the next section.

2. Main theorem

Theorem 2.1. *Let S be a right amenable semigroup with neutral element* 1. *Suppose $f, g, h, k \colon S \to \mathbb{C}$, $\varepsilon \ge 0$ and*

$$|f(xy) - g(x)h(y) - k(y)| \le \varepsilon, \quad x, y \in S. \tag{2.1}$$

Then there exist $F, G, H, K \colon S \to \mathbb{C}$ satisfying

$$F(xy) = G(x)H(y) + K(y), \quad x, y \in S, \tag{2.2}$$

so that the differences $f - F$, $g - G$, $h - H$ and $k - K$ are bounded.

We start with two lemmas.

Lemma 2.2. *Suppose that S is a semigroup, $\varphi, \psi, \xi \colon S \to \mathbb{C}$, $\delta \ge 0$, φ is unbounded and*

$$|\varphi(xy) - \varphi(x)\psi(y) - \xi(y)| \le \delta, \quad x, y \in S. \tag{2.3}$$

Then

$$\psi(xy) = \psi(x)\psi(y), \quad x, y \in S.$$

Proof. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence such that

$$0 \ne |\varphi(z_n)| \overset{n\to\infty}{\longrightarrow} \infty.$$

Using (2.3) with $x = z_n$ and dividing the obtained inequality by $|\varphi(z_n)|$, side by side, we have

$$\left| \frac{\varphi(z_n y)}{\varphi(z_n)} - \psi(y) - \frac{\xi(y)}{\varphi(z_n)} \right| \le \frac{\delta}{|\varphi(z_n)|}, \quad n \in \mathbb{N}, y \in S.$$

Letting $n \to \infty$ we obtain

$$\psi(y) = \lim_{n\to\infty} \frac{\varphi(z_n y)}{\varphi(z_n)}, \quad y \in S. \tag{2.4}$$

Using (2.3) with $z_n x$ instead of x and dividing the obtained inequality by $|\varphi(z_n)|$, side by side, we have

$$\left| \frac{\varphi(z_n xy)}{\varphi(z_n)} - \frac{\varphi(z_n x)}{\varphi(z_n)}\psi(y) - \frac{\xi(y)}{\varphi(z_n)} \right| \le \frac{\delta}{|\varphi(z_n)|}, \quad n \in \mathbb{N}, x, y \in S.$$

Passing with n to infinity and taking (2.4) into account we infer that

$$|\psi(xy) - \psi(x)\psi(y) - 0| \le 0, \quad x, y \in S,$$

which completes the proof. □

Lemma 2.3. *Suppose that S is a right amenable semigroup, $\varphi, \psi, \xi\colon S \to \mathbb{C}$ and $\delta \geq 0$. If*

$$|\varphi(xy) - \varphi(x)\psi(y) - \xi(y)| \leq \delta, \quad x, y \in S, \tag{2.5}$$

and

$$\psi(xy) = \psi(x)\psi(y), \quad x, y \in S,$$

then there exists a function $\eta\colon S \to \mathbb{C}$ such that

$$\eta(xy) = \eta(x)\psi(y) + \eta(y), \quad x, y \in S,$$

and

$$|\eta(x) - \xi(x)| \leq 2\delta, \quad x \in S. \tag{2.6}$$

Proof. Let M be a right invariant mean on $\mathcal{B}(S)$. By (2.5) we infer that, for every $y \in S$, the mapping

$$S \ni x \mapsto \varphi(xy) - \varphi(x)\psi(y)$$

is bounded. Hence we can define $\eta\colon S \to \mathbb{C}$ by the formula

$$\eta(y) := M_z(\varphi(zy) - \varphi(z)\psi(y)), \quad y \in S.$$

For $x, y \in S$ we have

$$\begin{aligned}
\eta(xy) &= M_z(\varphi(zxy) - \varphi(z)\psi(xy)) \\
&= M_z(\varphi(zxy) - \varphi(zx)\psi(y) + \varphi(zx)\psi(y) - \varphi(z)\psi(xy)) \\
&= M_z(\varphi(zxy) - \varphi(zx)\psi(y) + (\varphi(zx) - \varphi(z)\psi(x))\psi(y)) \\
&= M_z(\varphi(zxy) - \varphi(zx)\psi(y)) + M_z(\varphi(zx) - \varphi(z)\psi(x))\psi(y) \\
&= \eta(y) + \eta(x)\psi(y).
\end{aligned}$$

Moreover, since $\|M\| \leq 2$ (cf.(1.2)), for $y \in S$ we have

$$\begin{aligned}
\mid \eta(y) - \xi(y) \mid &= \mid M_z(\varphi(zy) - \varphi(z)\psi(y)) - \xi(y) \mid \\
&= \mid M_z(\varphi(zy) - \varphi(z)\psi(y)) - M_z(\xi(y)) \mid \\
&= \mid M_z(\varphi(zy) - \varphi(z)\psi(y) - \xi(y)) \mid \\
&\leq 2 \sup_{z \in S} \mid \varphi(zy) - \varphi(z)\psi(y) - \xi(y) \mid,
\end{aligned}$$

which jointly with (2.5) gives (2.6) and completes the proof. $\square$

Proof of Theorem 2.1. From (2.1) we get

$$|f(x) - g(x)h(1) - k(1)| \leq \varepsilon, \quad x \in S. \tag{2.7}$$

By (2.7), applied for xy, and (2.1) we obtain

$$|g(xy)h(1) + k(1) - g(x)h(y) - k(y)| \leq 2\varepsilon, \quad x, y \in S. \tag{2.8}$$

Let us consider the following cases.

1. $h = 0$.

Then, by (2.8), k is bounded, and by (2.7), so is f. Functions $F = 0$, $G = g$, $H = 0$ and $K = 0$ are as required.

2. $h(1) = 0$ and there is an $x_0 \in S$ with $h(x_0) \neq 0$.

By (2.8) with $y = x_0$ we see that g is bounded, and therefore, by (2.7), f is also bounded.

- If $g(x) = a \in \mathbb{C}$ is constant, then putting $F(x) = k(1)$, $G(x) = a$, $H = h$ and $K(x) = -ah(x) + k(1)$, we have

$$F(xy) = k(1) = G(x)H(y) + K(y), \quad x, y \in S,$$

and, by (2.8),

$$|K(x) - k(x)| \leq 2\varepsilon, \quad x \in S.$$

- If there are $x_1, x_2 \in S$ with $g(x_1) =: a \neq b := g(x_2)$ then, by (2.8) applied for x_1 and x_2 in place of x, we have

$$|ah(y) + k(y) - k(1)| \leq 2\varepsilon, \quad y \in S, \tag{2.9}$$

and

$$|bh(y) + k(y) - k(1)| \leq 2\varepsilon, \quad y \in S,$$

respectively. We conclude that

$$|a - b|\,|h(y)| \leq 4\varepsilon, \quad y \in S,$$

hence h is bounded. Now, by (2.9), k is also bounded. With $F = G = H = K = 0$ we get what is required.

3. $h(1) \neq 0$.

We put

$$h_1(x) := \frac{h(x)}{h(1)}, \;\; k_1(x) := \frac{k(x) - k(1)}{h(1)}, \quad x \in S.$$

After dividing both sides of (2.8) by $|h(1)|$ we get

$$|g(xy) - g(x)h_1(y) - k_1(y)| \leq \frac{2\varepsilon}{|h(1)|} =: \delta, \quad x, y \in S. \tag{2.10}$$

3.1. g and k_1 are bounded.

By (2.7) f is also bounded, so it is enough to put $F = 0$, $G = 0$, $H = h$ and $K = 0$ to get the assertion of the Theorem.

3.2. g is bounded and k_1 is not.

By (2.7) f is also bounded ($M_1 := \sup_{x \in S} |f(x)|$), and by (2.10) we infer that h_1 is unbounded. Let $M_2 > 0$ be such that $|g(x)| \leq M_2$ for $x \in S$ and let $(y_n)_{n \in \mathbb{N}}$ be a sequence such that $0 \neq |h_1(y_n)| \xrightarrow{n \to \infty} \infty$. By (2.10), for any $x_1, x_2 \in S$, we obtain

$$|g(x_1)h_1(y_n) + k_1(y_n)| \leq \delta + M_2, \quad n \in \mathbb{N},$$

and

$$|g(x_2)h_1(y_n) + k_1(y_n)| \le \delta + M_2, \quad n \in \mathbb{N},$$

so

$$|g(x_1) - g(x_2)|\,|h_1(y_n)| \le 2\delta + 2M_2, \quad n \in \mathbb{N}.$$

Thereby, g is constant ($g(x) =: a$ for $x \in S$). We put $F = 0$, $G = g$, $H = h$ and $K(x) = -ah(x)$, $x \in S$. It is obvious that (2.2) is satisfied. In order to finish the proof in this case it is enough to check that the difference $k - K$ is bounded. By (2.1) we have

$$|k(x) - K(x)| = |ah(x) + k(x)| \le \varepsilon + M_1, \quad x \in S.$$

3.3. g is unbounded.
By Lemma 2.2 we infer that

$$h_1(xy) = h_1(x)h_1(y), \quad x, y \in S. \tag{2.11}$$

On account of Lemma 2.3 there exists $k_2 \colon S \to \mathbb{C}$ such that

$$k_2(xy) = k_2(x)h_1(y) + k_2(y), \quad x, y \in S, \tag{2.12}$$

and

$$|k_2(x) - k_1(x)| \le 2\delta, \quad x \in S.$$

We define

$$\begin{aligned}
F(x) &= [k_2(x) + g(1)h_1(x)]h(1),\\
G(x) &= k_2(x) + g(1)h_1(x),\\
H(x) &= h_1(x)h(1) = h(x),\\
K(x) &= k_2(x)h(1), \quad x \in S.
\end{aligned}$$

It easy to check, using (2.11) and (2.12), that (2.2) is fulfilled. Moreover, we have

$$\begin{aligned}
|k(y) - K(y)| &= |k(y) - k_2(y)h(1)|\\
&\le |k(y) - k_1(y)h(1)| + |k_1(y)h(1) - k_2(y)h(1)|\\
&\le |k(1)| + 2\delta\,|h(1)|, \quad y \in S,
\end{aligned}$$

$$\begin{aligned}
|g(x) - G(x)| &= |g(x) - k_2(x) - g(1)h_1(x)|\\
&\le |g(x) - g(1)h_1(x) - k_1(x)| + |k_1(x) - k_2(x)|\\
&\le \delta + 2\delta, \quad x \in S,
\end{aligned}$$

$$\begin{aligned}
|f(x) - F(x)| &= |f(x) - G(x)h(1)|\\
&\le |f(x) - g(x)h(1) - k(1)| + |k(1)| + |g(x) - G(x)|\,|h(1)|\\
&\le \varepsilon + |k(1)| + 3\delta\,|h(1)|, \quad x \in S.
\end{aligned}$$

□

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References

[1] Aczél, J.: Vorlesungen über Funktionalgleichungen und ihre Anwendungen. Birkhäuser Verlag Basel, 1961. English edition: Lectures on Functional Equations and their Applications, Academic Press, New York (1966)
[2] Aczél, J.: On Applications and Theory of Functional Equations. Birkhäuser, Basel (1969)
[3] Aczél, J., Dhombres, J.: Functional Equations in Several Variables. Cambridge University Press, Cambridge (1989)
[4] Ebanks, B.: General solution of a simple Levi–Civitá functional equation on non-abelian groups. Aequationes Math. **85**(3), 359–378 (2013)
[5] Járai, A., Székelyhidi, L.: Regularization and general methods in the theory of functional equations. Aequationes Math. **52**(1), 10–29 (1996)
[6] McKiernan, M.A.: Equations of the form $H(x \circ y) = \sum_i f_i(x)g_i(y)$. Aequationes Math. **16**(1–2), 51–58 (1977)
[7] Shulman, E.: Functional equations of homological type, Ph.D. Thesis, Moscow Institute of Electronics and Mathematics, Moscow 1994 (Russian)
[8] Shulman, E.: Group representations and stability of functional equations. J. London Math. Soc. **52**(2), 111–120 (1996)
[9] Stetkaer, H.: Functional equations and matrix-valued spherical functions. Aequationes Math. **69**(3), 271–292 (2005)
[10] Székelyhidi, L.: On the Levi-Civita functional equation. Ber. Math.-Statist. Sekt. Forsch. Graz, vol. 301, pp. 23 (1988)
[11] Székelyhidi, L.: Stability properties of functional equations describing the scientific laws. J. Math. Anal. Appl. **150**, 151–158 (1990)
[12] Vincze, E.: Eine allgemeinere Methode in der Theorie der Funktionalgleichungen I. Publ. Math. Debrecen **9**, 149–163 (1963)

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